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A variational principle for symplectic connections

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Abstract

We introduce a variational principle for symplectic connections and study the corresponding field equations. For two-dimensional compact symplectic manifolds we determine all solutions of the field equations. For two-dimensional non-compact simply connected symplectic manifolds we give an essentially exhaustive list of solutions of the field equations. Finally we indicate how to construct from solutions of the field equations on (M, ω) solutions of the field equations on the cotangent bundle to M with its standard symplectic structure. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

On a Riemannian manifold (M, g) there exists a unique preferred linear connection ∇ – called the Levi Civita connection – characterized by having no torsion and by the fact that g is parallel. On a symplectic manifold (M, ω) there exist preferred linear connections ∇ – called symplectic connections – characterized by having no torsion and by the fact that ω is parallel. Such connections are never unique: the set of such connections can be identified (in a non-canonical way) to the space of completely symmetric covariant tensor fields of order 3 on M.

If one believes that linear connections are a useful tool to do geometry, it seems natural to introduce a selection rule – a variational principle – to choose one (or a restricted class)

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of symplectic connections. It may of course happen that the field equations associated with this variational principle do not admit solutions on certain symplectic manifolds; we have for the moment no idea as to a possible answer to such a question. But even if this would be the case, the determination of those symplectic manifolds for which the space of solutions of the field equations is not empty, is in our opinion an interesting problem. It seems to be even more instructive to determine the structure of the space of equivalence classes of solutions of the field equations (equivalence class under the action of the diffeomorphism group).

In Section 2 we recall elementary properties of symplectic connections of the corresponding curvature tensor and determine all polynomial invariants in the curvature which are of degree less than or equal to 2. In Section 3 we introduce two variational principles for a symplectic connection and prove that they lead to the same field equations; this slightly surprising fact is due to a nice identity relating, on a symplectic vector space (V_0, ω_0), the exterior product of 2-forms to the natural pairing of these 2-forms induced by ω_0. Section 4 describes classes of solutions of the field equations which one can read in the literature (this is true for certain type of symplectic manifolds only). In Section 5 we begin the study of two-dimensional manifolds (M, ω) admitting a preferred symplectic connection (i.e. a connection solution of our field equations). The essential point here is the introduction of a function β which controls completely the geometry. In Section 6 we prove that on a compact symplectic manifold the function β is either identically zero or that $\mathrm{d}\beta$ is not identically zero. In this second case we prove that the Hamiltonian vector field X_β associated to β generates a S^1 action on (M, ω). From this, one deduces easily that if the genus of M is such that $g \geq 1$, the preferred connection must be locally symmetric. Section 7 is devoted to the sphere case. The function β admits necessarily only two isolated non-degenerate critical points; one constructs on the sphere minus these two points the most general solution of the field equations and one shows that it cannot be extended to the whole of S^2. In Section 8 we give a list of the symmetric symplectic surfaces and in Section 9 we give a list of the compact locally symmetric surfaces. It turns out that in most of the compact cases the locally symmetric connection is associated to a Riemannian locally symmetric space. In Section 10 we prove that on each compact symplectic surface (M, ω) there exists a preferred connection and we characterize the space of equivalence classes of such connections. Section 11 gives a local description of the preferred connections on the plane $(\mathbb{R}^2, \omega_0)$; it also gives a list of geodesically complete such connections. It appears from this list that the situation is immensely more complicated in the non-compact case. Finally in Section 12 we show how to lift preferred connections from (M, ω) to the cotangent bundle T^*M with its standard symplectic structure.

2. Symplectic connections: symplectic curvature tensor

Let (M, ω) be a symplectic manifold. A linear connection ∇ on (M, ω) is said to be symplectic iff: (i) T^∇ (= torsion of ∇) $= 0$; (ii) $\nabla\omega = 0$. We recall the proof of the following classical proposition.

Proposition 2.1. *Let (M, ω) be a symplectic manifold. The set of symplectic connections on (M, ω) can be identified with the affine space of completely symmetric tensor fields of type $\binom{0}{3}$ on M.*

Proof. Let $\overline{\nabla}$ be any torsion-free connection on (M, ω); any other torsion-free linear connection can be expressed as

$$\nabla_X Y = \overline{\nabla}_X Y + S(X, Y),$$

where X, Y are vector fields and S is symmetric. The condition $\nabla\omega = 0$ reads

$$(\overline{\nabla}_X \omega)(Y, Z) - \underline{S}(X, Y, Z) + \underline{S}(X, Z, Y) = 0,$$

where $\underline{S}(X, Y, Z) = \omega(S(X, Y), Z)$. A particular solution to this equation is given by

$$\underline{S}(X, Y, Z) = \tfrac{1}{3}(\overline{\nabla}_X \omega(Y, Z) + \overline{\nabla}_Y \omega(X, Z)).$$

Any other solution $\underline{S}'$ differs from this one by a $\underline{\widetilde{S}}$ which is completely symmetric. □

The curvature R of a symplectic connection ∇ on (M, ω) is defined as usual by

$$R(X, Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z.$$

To this $\binom{1}{3}$ tensor field one can associate a $\binom{0}{4}$ tensor field $\underline{R}$ by

$$\underline{R}(X, Y, Z, T) = \omega(R(X, Y)Z, T).$$

This will be called the curvature tensor of ∇.

The following proposition is obvious.

Proposition 2.2. *Let $\underline{R}$ be the curvature tensor of a symplectic connection ∇ on (M, ω). Then*

(i) $\underline{R}(X, Y, Z, T) = -\underline{R}(Y, X, Z, T) = \underline{R}(X, Y, T, Z)$,
(ii) $\oint_{X,Y,Z} \underline{R}(X, Y, Z, T) = 0$,
(iii) $\oint_{X,Y,Z} (\nabla_X \underline{R})(Y, Z, T, U) = 0$.

The Ricci tensor of the symplectic connection ∇ on (M, ω) is defined by

$$r(X, Y) = \mathrm{tr}(Z \to R(X, Z)Y).$$

The following properties are easy to check.

Proposition 2.3. *Let r be the Ricci tensor of a symplectic connection ∇ on (M, ω). Then*

(i) $r(X, Y) = r(Y, X)$,
(ii) *in a local coordinate system* $\{x^a\ ;\ 1 \le a \le 2n = \dim M\}$

$$r_{ab} \overset{\text{not}}{=} r(\partial_a, \partial_b) = \underline{R}_{cabd}\,\omega^{cd},$$

if $\underline{R}_{cdab}$ are the components of the curvature tensor of ∇ and if

$$\omega^{cd}\omega_{dp} = \delta^c_p,$$

(iii) *in a local coordinate system*

$$r_{ab} = \tfrac{1}{2}\underline{R}_{cdab}\omega^{cd},$$

(iv) *define the tensor field E on (M, ω) by the expression in any local coordinate system*

$$E_{abcd} = \frac{-1}{2(1+n)}[2\omega_{ab}r_{cd} + \omega_{ac}r_{bd} + \omega_{ad}r_{bc} - \omega_{bc}r_{ad} - \omega_{bd}r_{ac}],$$

where $2n = \dim M$. *Then*

$$E_{abcd} = -E_{bacd} = E_{abdc}, \qquad \oint_{a,b,c} E_{abcd} = 0,$$

$$\omega^{ad}E_{abcd} = r_{bc}, \qquad \omega^{ab}E_{abcd} = 2r_{cd},$$

(v) *let W be the tensor field on (M, ω) defined by*

$$\underline{R} = E + W.$$

Then W has all symmetries of the curvature tensor. In dimension 2 ($n = 1$), $W = 0$, hence $\underline{R} = E$. Furthermore

$$\omega^{ad}W_{abcd} = \omega^{ab}W_{abcd} = 0.$$

We observe that in view of the above proposition and of the symmetry of the Ricci tensor, there is no analog of the scalar curvature for a symplectic connection.

It is natural to define *symplectic Einstein manifolds* (SE) as symplectic manifolds admitting a symplectic connection ∇ such that

$$\underline{R} = W,$$

or equivalently

$$r = 0.$$

Similarly we define *symplectic simple manifolds* (SS) as symplectic manifolds admitting a symplectic connection ∇ such that

$$\underline{R} = E.$$

The following lemma, which is a direct consequence of the definition of the tensor field E, gives the list of polynomial invariants in the curvature tensor of degree smaller or equal to 2.

Lemma 2.1. *If $\underline{R}_{abcd}$ (resp, r_{ab}) are the components of the curvature tensor (resp. the Ricci tensor) of a symplectic connection ∇ on (M, ω) denote*

$$\underline{R}^{abcd} = \omega^{aa'}\omega^{bb'}\omega^{cc'}\omega^{dd'}\underline{R}_{a'b'c'd'}, \qquad r^{ab} = \omega^{aa'}\omega^{bb'}r_{a'b'}.$$

Then

$$\underline{R}_{abcd}\underline{R}^{abcd} = -\frac{4}{1+n} r_{ab} r^{ab} + W_{abcd} W^{abcd}.$$

Proposition 2.4. *Any polynomial invariant of degree smaller or equal to 2, in the curvature tensor $\underline{R}$ of a symplectic connection ∇ on (M, ω), is a linear combination of the two following invariants*

$$r_{ab} r^{ab} \qquad \underline{R}_{abcd}\underline{R}^{abcd}.$$

3. A variational principle for symplectic connections

An invariant functional J on the space of symplectic connections ∇ on (M, ω) may be defined by an expression of the form

$$J = \int A \frac{\omega^n}{n!},$$

where A is a scalar invariant depending on the curvature $\underline{R}$, eventually of a certain number of its covariant derivatives, and of the symplectic form ω.

We have assumed in this paper that A is a polynomial function in $\underline{R}$ of degree smaller or equal to 2. In view of Proposition 2.4, any functional is a linear combination of the two following ones:

$$J_1 \stackrel{\text{def}}{=} \frac{1}{2} \int r_{ab} r^{ab} \frac{\omega^n}{n!}, \qquad J_2 \stackrel{\text{def}}{=} \int \underline{R}_{abcd}\underline{R}^{abcd} \frac{\omega^n}{n!}.$$

The Euler–Lagrange equations for J_1 read:

$$\oint_{X,Y,Z} (\nabla_X r)(Y, Z) = 0 \quad \forall X, Y, Z \text{ vector fields on } M.$$

Locally this is expressed as

$$\oint_{a,b,c} r_{ab;c} = 0, \tag{3.1}$$

where ; means covariant derivative.

The Euler–Lagrange equations for J_2 read locally:

$$\oint_{b,c,d} \underline{R}_{pbcd;q}\,\omega^{pq} = 0. \tag{3.2}$$

Using Bianchi identities (Proposition 2.2, (iii)) appropriately contracted one gets

Proposition 3.1. *The Euler–Lagrange equations* (3.1) *and the Euler–Lagrange equations* (3.2) *coincide.*

This can be understood using the following lemma, due to Rawnsley.

Lemma 3.1. *Let* $(\mathbb{R}^{2n}, \omega_0)$ *be the standard symplectic vector space and let* φ, ψ *be elements of* $\bigwedge^2(\mathbb{R}^{2n})$. *If* $\{e_a\ ,\ 1 \le a \le 2n\}$ *is a basis of* $\mathbb{R}^{2n}$ *and* $\{e_*^a\}$ *is the dual basis of* $\mathbb{R}_*^{2n}$, *we have for* φ, ψ, ω_0 *the representation*

$$\varphi = \tfrac{1}{2}\varphi_{ab} e_*^a \wedge e_*^b, \qquad \psi = \tfrac{1}{2}\psi_{ab} e_*^a \wedge e_*^b, \qquad \omega_0 = \tfrac{1}{2}\omega_{ab} e_*^a \wedge e_*^b.$$

Define a bilinear symmetric form B *on* $\bigwedge^2(\mathbb{R}^{2n})$ *by*

$$B(\varphi, \psi) = \tfrac{1}{2}\varphi_{ab}\psi_{a'b'}\omega^{aa'}\omega^{bb'}, \qquad \omega^{aa'}\omega_{a'b} = \delta_b^a.$$

Then.

$$\varphi \wedge \psi \wedge \omega_0^{n-2} = \tfrac{1}{2}[B(\varphi, \omega_0)B(\psi, \omega_0) - B(\varphi, \psi)]\omega_0^n.$$

The proof is by straightforward calculation.

The curvature R at a point is a 2-form with values in the endomorphisms of the tangent space at this point; one may define $R \overset{\wedge}{\circ} R$ as the 4-form with values in the endomorphisms of the tangent space obtained by taking the exterior product of the 2-forms and composing the endomorphisms. The 4-form

$$\operatorname{tr}(R \overset{\wedge}{\circ} R)$$

at the given point is then scalar valued. Similarly one may compose the endomorphisms $B(R, \omega_0)$ and $B(R, \omega_0)$ and take the trace; and do the same for $B(R, R)$. Lemma 3.1 above then implies that

$$P_1 \wedge \omega^{n-2} = \frac{1}{16\pi^2}\left[r_{ab}r^{ab} - \frac{1}{2}\underline{R}_{abcd}\underline{R}^{abcd}\right]\omega^n,$$

where P_1 is a 4-form which represents the 1st Pontrjagin class of M. As the Pontrjagin class does not depend on the chosen connection, we have, in view of Proposition 2.4

Proposition 3.2. *A variational principle for symplectic connections whose lagrangian density is a polynomial in the curvature of degree smaller or equal to 2 is unique.*

4. Examples

Let (M, ω) be a symplectic manifold. An almost complex structure J is said to be *compatible* with ω if

(i) $\omega(JX, JY) = \omega(X, Y)$, $\quad\forall X, Y$ vector fields,

(ii) $\omega(X, JX) > 0$ if $X \neq 0$.

Such compatible almost complex structure always exist [1]. Define, for a given compatible J, a Riemannian structure g by

$$g(X, Y) = \omega(X, JY),$$

and let $\nabla_{(g)}$ be the Levi Civita connection associated to g. The manifold (M, ω, J) is said to be *Kähler* if

$$\nabla_{(g)} J = 0.$$

Then $\nabla_{(g)}\omega = 0$ and $\nabla_{(g)}$ is a symplectic connection. There are now two curvature tensors:
(i) $\underline{R}(X, Y, Z, T) = \omega(R_{(g)}(X, Y)Z, T)$ which is the symplectic one.
(ii) $\underline{R}'(X, Y, Z, T) = g(R_{(g)}(X, Y)Z, T)$ which is the Riemannian one.
They are related by

$$\underline{R}'(X, Y, Z, T) = \underline{R}(X, Y, Z, JT).$$

There is only one Ricci tensor. There is also a Ricci form

$$\rho(X, Y) = r(X, JY).$$

In a local coordinate system one gets

$$\begin{aligned}\rho_{ab} = r_{ap} J_b^p &= \tfrac{1}{2}\underline{R}_{stap}\omega^{st} J_b^p \\ &= \tfrac{1}{2} R_{sta}{}^{l}\omega_{lp}\omega^{st} J_b^p \\ &= \tfrac{1}{2}\underline{R}'_{stab}\omega^{st} = \tfrac{1}{2}\underline{R}'_{abst}\omega^{st}.\end{aligned}$$

Hence by the Bianchi identities, in the Riemannian case, one gets

$$\oint_{a,b,c} \rho_{ab;c} = 0,$$

which means that ρ is a closed 2-form.

The field equations

$$\oint_{X,Y,Z} (\nabla_X r)(Y, Z) = 0$$

are equivalent to

$$(\nabla_X r)(X, X) = 0 \qquad \forall X \text{ vector field.}$$

In the Kähler case, this equation is known as the D'Atri condition [2, Section 16.53, p. 450]. The D'Atri condition is satisfied if the geodesic symmetry at a point preserves the volume (locally) [2, Section 16.52, p. 450]. Clearly in dimension ≥ 4 Kähler Einstein spaces (i.e. $r = \lambda g$) are solutions of the field equations. In particular, any coadjoint orbit of a compact group admits such a Kähler Einstein structure which is unique (up to homothety) [2, Theorem 8.2, p. 208]. It is a famous result that any compact complex manifold with negative first Chern class admits a Kähler Einstein metric [2, Theorem 11.17, p. 322]. Also the $K3$ surfaces admit Kähler Einstein metrics [2, Section 12.105, p. 365].

5. The two-dimensional case: a local approach I

We first observe that in dimension 2, the symplectic curvature tensor $\underline{R}$ reduces to its E part; also any covariant antisymmetric 2-tensor field is proportional to ω. So in any local chart one has

$$\underline{R}_{abcd} = -\omega_{ab} r_{cd}. \tag{5.1}$$

Also the field equations, in dimension 2, are equivalent to the existence of a 1-form u such that

$$r_{ab;c} \overset{\text{not}}{=} (\nabla_c r)(\partial_a, \partial_b) = \omega_{ac} u_b + \omega_{bc} u_a. \tag{5.2}$$

Lemma 5.1. *There exists a function β such that the covariant derivative of the 1-form u is given by*

$$u_{a;b} = \beta\omega_{ab}. \tag{5.3}$$

The second covariant derivative of the Ricci tensor is expressed in terms of this function β by

$$r_{ab;cd} = \beta(\omega_{ab}\omega_{cd} + \omega_{bc}\omega_{ad}). \tag{5.4}$$

The first and second covariant derivatives of β are given by

$$\beta_a = r_a^k u_k, \tag{5.5}$$

$$\beta_{a;b} = -u_a u_b + \beta r_{ab}. \tag{5.6}$$

Proof. The Ricci identity for the second covariant derivatives of the Ricci tensor reads

$$r_{ab;cd} - r_{ab;dc} = -R_{dca}{}^p r_{pb} - R_{dcb}{}^p r_{ap} = \omega_{dc}(r_a^p r_{pb} + r_b^p r_{pa}) = 0.$$

Hence

$$\omega_{ac} u_{b;d} + \omega_{bc} u_{a;d} - \omega_{ad} u_{b;c} - \omega_{bd} u_{a;c} = 0, \qquad 2(u_{b;a} + u_{a;b}) = 0,$$

which proves (5.3). Formula (5.4) is a direct consequence.

The Ricci identity for the second covariant derivative of the 1-form u gives

$$u_{a;bc} - u_{a;cb} = -R_{cba}{}^k u_k = \omega_{cb} r_a^k u_k = \omega_{ab}\beta_c - \omega_{ac}\beta_b.$$

Hence (5.5). Deriving this relation we get

$$\beta_{a;b} = r_{ap;b}\omega^{pk} u_k + r_{ap}\omega^{pk}\beta\omega_{kb} = \omega_{pb} u_a \omega^{pk} u_k + \beta r_{ab} = -u_a u_b + \beta r_{ab}. \qquad \square$$

The function β will play the crucial role in all that follows. In particular:

Proposition 5.1. *The symplectic connection ∇ is locally symmetric if and only if $\beta = 0$.*

Proof. If $\beta = 0$, (5.6) implies $u = 0$ and hence $\nabla r = 0$; thus $\nabla R = 0$. Conversely if $\nabla R = 0$, $u = 0$ and (5.3) implies $\beta = 0$.

Lemma 5.2. *There exist real numbers A, B such that*

$$r_{ab}\overline{u}^a\overline{u}^b = \beta^2 + B, \tag{5.7}$$

$$\tfrac{1}{4}r_{ab}r^{ab} = \beta + A, \tag{5.8}$$

where $\overline{u}^a\omega_{ab} = u_b$.

Proof. Deriving the left-hand side of (5.7):

$$(r_{ab}\overline{u}^a\overline{u}^b)_{;c} = r^{ab}\beta(\omega_{ac}u_b + u_a\omega_{bc}) = 2\beta\beta_c = (\beta^2)_{;c}.$$

Hence formula (5.7).

Deriving the left-hand side of (5.8):

$$\tfrac{1}{4}(r_{ab}r^{ab})_{;c} = \tfrac{1}{2}(\omega_{ac}u_b + \omega_{bc}u_a)r^{ab} = r_c^b u_b = \beta_c.$$

Hence formula (5.8).

Lemma 5.3.

(i) *The Hamiltonian vector field X_β and the vector field $\overline{u}$ ($i(\overline{u})\omega = u$) commute.*

(ii) *The 1-forms u and $\mathrm{d}\beta$ are linearly independent at all points p where $r_p(\overline{u}, \overline{u}) \neq 0$. In fact*

$$u \wedge \mathrm{d}\beta = -r(\overline{u}, \overline{u})\omega. \tag{5.9}$$

Proof. One has

$$i([\overline{u}, X_\beta])\omega = (i(\overline{u})\mathcal{L}_{X_\beta} - \mathcal{L}_{X_\beta}i(\overline{u}))\omega, = -i(X_\beta)\,\mathrm{d}u - \mathrm{d}u(X_\beta).$$

But Eq. (5.3) implies

$$\mathrm{d}u = -2\beta\omega$$

and

$$u(X_\beta) = \omega_{ab}\overline{u}^a X_\beta^b = -\overline{u}^a\beta_a = r_{ab}\overline{u}^a\overline{u}^b = \beta^2 + B.$$

Hence

$$i([\overline{u}, X_\beta])\omega = 2\beta\,\mathrm{d}\beta - 2\beta\,\mathrm{d}\beta = 0,$$

which proves (i). For (ii) notice that

$$\begin{aligned} u \wedge \mathrm{d}\beta &\overset{\mathrm{loc}}{=} (u_1\beta_2 - u_2\beta_1)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \\ &= -\omega_{12}\omega^{ab}u_a\beta_b\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \\ &= \overline{u}^p\beta_p\omega = -r(\overline{u}, \overline{u})\omega. \qquad \square \end{aligned}$$

Proposition 5.2. *Assume $\mathrm{d}\beta \neq 0$ and let $U = \{p \in M \mid r_p(\overline{u}, \overline{u}) \neq 0\}$. Then the preferred connection must be given locally on U by the formulae*

$$\nabla_{\overline{u}}\overline{u} = -\beta\overline{u}, \qquad \nabla_{\overline{u}}X_\beta = -\beta X_\beta, \qquad \nabla_{X_\beta}X_\beta = (\beta^2 + 2A\beta - B)\overline{u}. \tag{5.10}$$

Conversely, given a 1-form u and a non-constant function β such that $u(X_\beta) = \beta^2 + B$, then formula (5.10) define on the open set $\beta^2 \neq -B$ a preferred connection.

Proof. One has locally

$$\nabla_{\overline{u}}\overline{u} = \overline{u}^a \overline{u}^b_{;a}\partial_b = \overline{u}^a \omega^{pb}\beta\omega_{pa}\partial_b = -\beta\overline{u}.$$

Similarly

$$\nabla_{\overline{u}}X_\beta = \overline{u}^a \beta^b_{;a}\partial_b = \overline{u}^a\omega^{pb}(-u_p u_a + \beta r_{pa})\partial_b = -\beta X_\beta.$$

Finally

$$\begin{aligned}
\nabla_{X_\beta}X_\beta &= \beta^a\beta^b_{;a}\partial_b = \beta_p\omega^{pa}\omega^{qb}(-u_q u_a + \beta r_{qa})\partial_b \\
&= \overline{u}^p\beta_p\overline{u} - \beta r^p_q r^s_p u_s \omega^{qb}\partial_b \\
&= -(\beta^2 + B)\overline{u} - \beta r^p_q r_{pl}\omega^{ls}u_s\omega^{qb}\partial_b \\
&= -(\beta^2 + B)\overline{u} + \tfrac{1}{2}\beta(r_{cd}r^{cd})\omega_{ql}\omega^{ls}u_s\omega^{qb}\partial_b \\
&= -(\beta^2 + B)\overline{u} + \tfrac{1}{2}\beta(r_{cd}r^{cd})\overline{u}, \\
&= (-\beta^2 - B + 2\beta^2 + 2A\beta)\overline{u} \\
&= (\beta^2 + 2A\beta - B)\overline{u}.
\end{aligned}$$

Conversely assuming (5.10) we get

$$R(\overline{u}, X_\beta)\overline{u} = -\overline{u}(\beta)X_\beta = (\beta^2 + \beta)X_\beta, \quad R(\overline{u}, X_\beta)X_\beta = -2(\beta + A)(\beta^2 + B)\overline{u},$$

and for the Ricci tensor

$$\begin{aligned}
r(\overline{u}, \overline{u}) &= \beta^2 + B, \qquad r(\overline{u}, X_\beta) = 0, \\
r(X_\beta, X_\beta) &= 2(\beta + A)(\beta^2 + B).
\end{aligned}$$

The non-vanishing components of the covariant derivative of the Ricci tensor are

$$(\nabla_{\overline{u}}r)(X_\beta, X_\beta) = -2(\beta^2 + B)^2, \qquad (\nabla_{X_\beta}r)(\overline{u}, X_\beta) = (\beta^2 + B)^2,$$

which shows that the field equations are indeed satisfied. One checks also that the constructed connection is symplectic. □

6. The two-dimensional case: a global approach I

Lemma 6.1. *If $\mathrm{d}\beta = 0$ and $\beta \neq 0$, the manifold M is not compact.*

Proof. One has $\omega = -(1/2\beta)\,\mathrm{d}u = \mathrm{d}(u/-2\beta)$ which cannot occur in the compact case. □

Lemma 6.2. *If (M, ω) is compact and ∇ is not locally symmetric then all critical points of β are either absolute maxima or absolute minima. If $K = \max\beta$ (resp. $k = \min\beta$) then $K > 0$, $k = -K$ and $B = -K^2 < 0$.*

Proof. One simply observes that at a critical point p,

$$u_p(X_\beta) = \beta^2(p) + B = 0.$$

Hence $B = -K^2$.

Lemma 6.3. *The function β admits at least one non-degenerate critical point. (M is assumed to be compact.)*

Proof. Let p be a critical point. Then

$$\begin{aligned}
\mathrm{Hess}_p\beta &= \det(\beta_{a;b})(p) = \beta_{1;1}\beta_{2;2} - \beta_{1;2}\beta_{2;1} \\
&= \det\omega\tfrac{1}{2}\omega^{aa'}\omega^{bb'}\beta_{a;b}\beta_{a';b'} \\
&= \det\omega\tfrac{1}{2}\omega^{aa'}\omega^{bb'}(-u_a u_b + \beta r_{ab})(-u_{a'}u_{b'} + \beta r_{a'b'}) \\
&= \det\omega\tfrac{1}{2}(-2\beta r(\overline{u},\overline{u}) + \beta^2 r_{ab}r^{ab}) \\
&= \det\omega(-\beta(\beta^2 + B) + \beta^2(2\beta + 2A)) \\
&= \det\omega\beta(p)(-\beta^2 - B + 2\beta^2 + 2A\beta) \\
&= \det\omega\beta(p)(\beta^2 + 2A\beta - B) \\
&= \det\omega(\pm 1)2K(K^2 \pm AK) \\
&= \det\omega(\pm 1)2K^2(K \pm A).
\end{aligned}$$

Hence if $A \neq \pm K$ all critical points are non-degenerate; if A has one of the exceptional values, then either all maximum points or all minimum points are non-degenerate. □

Lemma 6.4. *Let φ_t be the flow associated to the vector field $\overline{u}$ (which is complete as M is assumed compact). Then*

(i) *the fixed points of φ_t are critical points of β*
(ii) *any non-constant integral curve of $\overline{u}$ goes from the set of minima of β (for $t = -\infty$) to the set of maxima of β (for $t = +\infty$).*

Proof. If $\overline{u}_p = 0$, then $\beta_a(p) = r_a^k u_k(p) = 0$. Furthermore

$$\overline{u}^a\beta_a = \frac{\mathrm{d}\beta}{\mathrm{d}t} = -\beta^2 + K^2.$$

Hence

$$\beta(t) = K\frac{(K+\beta_0)\mathrm{e}^{2Kt} - (K-\beta_0)}{(K+\beta_0)\mathrm{e}^{2Kt} + (K-\beta_0)}. \quad \square$$

Proposition 6.1. *Assume M compact and $\mathrm{d}\beta \neq 0$. Let ψ_s be the flow associated to the hamiltonian vector field X_β. Then*

(i) *ψ_s is a group of affine transformations of (M, ∇).*
(ii) *There exists a smallest positive number τ such that $\psi_\tau = id$; in particular X_β generates a strongly Hamiltonian action of the torus S^1 on (M, ω).*

Proof. X_β is an affine vector field if

$$\mathcal{L}_{X_\beta}\nabla_Y Z = \nabla_{\mathcal{L}_{X_\beta}Y} Z + \nabla_Y \mathcal{L}_{X_\beta} Z$$

for any vector fields X, Y, Z. This is easily checked using for vectors Y and Z the vectors $\overline{u}$ and X_β, the relation (5.10) and Lemma 5.3(i).

As β admits an isolated critical point p, consider a neighborhood Ω_p, which is the domain of a chart with coordinates x, y such that in Ω_p

$$\beta = (\mp K \pm (x^2 + y^2)).$$

The upper signs correspond to p, a minimum, and the lower signs correspond to p, a maximum. Let $q \in \Omega_p$. The local curve $\beta = \beta(q)$ is a circle, which is an integral curve of X_β. Let τ be the smallest positive number such that $\psi_\tau(q) = q$. We claim that τ does not depend on q. If $q' \in \Omega_p$ and $\beta(q') = \beta(q)$, $\tau_q = \tau_{q'}$. If $\beta(q') \neq \beta(q)$, let l be defined by $\beta(\varphi_l(q')) = \beta(q)$. Then

$$\psi_\tau \circ \varphi_l(q') = \varphi_l(q') = \varphi_l \circ \psi_\tau(q')$$

and thus

$$\psi_\tau(q') = q'.$$

The diffeomorphism ψ_τ is thus the identity when reduced to ω_p; as it is an affine transformation $\psi_\tau = id$. The S^1 action is then defined by

$$e^{i\theta} \cdot x = \psi_{\tau\theta/2\pi}(x). \qquad \square$$

Theorem 6.1. *Let (M, ω) be a connected two-dimensional compact symplectic manifold such that its genus g is > 0. If ∇ is a symplectic connection which satisfies the field equations, then ∇ is locally symmetric.*

Proof. As ∇ is a solution of the field equations, the second covariant derivative of the Ricci tensor defines a function β. If $\beta = 0$, ∇ is locally symmetric. As M is compact, β cannot be constant and different from 0. If β is not constant, there exists a strongly Hamiltonian action of S^1. From this follows easily that β is a Morse Bott function with even dimensional critical manifolds [3]. There cannot be a two-dimensional submanifold as the action of S^1 is not trivial. Hence all critical submanifolds are of dimension 0 and hence isolated critical points, which are non-degenerate. This implies that

$$p + q = \chi(M),$$

where p is the number of minima and q the number of maxima. Also $p \geq 1$, $q \geq 1$ and $\chi(M) < 2$ lead to the desired contradiction. $\square$

Remarks.

(i) *The only case where a non-locally symmetric solution of the field equations in the compact case might exist is the sphere S^2.*

(ii) *As the Hessian is positive we have*

$$A > K > -A.$$

7. The sphere case

We know that if (M, ω) is compact, and ∇ is not locally symmetric, then (M, ω) is diffeomorphic to the standard sphere S^2, endowed with its standard symplectic structure ω_0 and the S^1 action is the usual rotation around an axis [4]. We shall scale ω_0 such that the area of S^2, relative to ω_0, is 4π.

We shall use "latitude, longitude" (θ, φ) coordinates on S^2 and write local expressions in the open set $S^2 \setminus \{\text{north and south pole}\}$. In these coordinates

$$\omega = \cos\theta \, \mathrm{d}\theta \wedge \mathrm{d}\varphi, \qquad g_0 = \mathrm{d}\theta \otimes \mathrm{d}\theta + \cos^2\theta \, \mathrm{d}\varphi \otimes \mathrm{d}\varphi,$$

where g_0 is the standard Riemannian metric on S^2. If $\overline{\nabla}$ denotes the Levi Civita connection corresponding to g_0, one has

$$\overline{\nabla}_{\partial_\theta} \partial_\theta = 0 \qquad \overline{\nabla}_{\partial_\varphi} \partial_\varphi = \sin\theta\cos\theta\,\partial_\theta,$$
$$\overline{\nabla}_{\partial_\theta} \partial_\varphi = -\tan\theta\,\partial_\varphi = \overline{\nabla}_{\partial_\varphi} \partial_\theta.$$

The function β associated to the symplectic connection reads

$$\beta = K \sin\theta, \qquad X_\beta = -K\partial_\varphi.$$

The symplectic conditions are:

$$\Gamma^\theta_{\theta\theta} + \Gamma^\varphi_{\theta\varphi} = -\tan\theta, \qquad \Gamma^\theta_{\theta\varphi} + \Gamma^\varphi_{\varphi\varphi} = 0.$$

If one writes the 1-form u as

$$u = u_1 \,\mathrm{d}\theta + u_2 \,\mathrm{d}\varphi$$

the vector field $\overline{u}$ is

$$\overline{u} = \frac{u_2}{\cos\theta}\partial_\theta - \frac{u_1}{\cos\theta}\partial_\varphi$$

and the condition $[\overline{u}, X_\beta] = 0$ tells us that

$$\partial_\varphi u_2 = 0 = \partial_\varphi u_1.$$

As X_β is an affine vector field, the Christoffel symbols are independent of φ. Recall that we have

$$u_{a;b} = \beta\omega_{ab}.$$

Hence

$$\partial_\theta u_1 - \Gamma^\theta_{\theta\theta} u_1 - \Gamma^\varphi_{\theta\theta} u_2 = 0, \qquad -\Gamma^\theta_{\theta\varphi} u_1 - \Gamma^\varphi_{\theta\varphi} u_2 = K\sin\theta\cos\theta,$$
$$\partial_\theta u_2 - \Gamma^\theta_{\theta\varphi} u_1 - \Gamma^\varphi_{\theta\varphi} u_2 = -K\sin\theta\cos\theta, \qquad -\Gamma^\theta_{\varphi\varphi} u_1 - \Gamma^\varphi_{\varphi\varphi} u_2 = 0.$$

The second and third of these equations imply

$$u_2 = K\cos^2\theta + a \quad (a \in \mathbb{R}).$$

On the other hand:

$$\overline{u}^a\beta_a = -r(\overline{u},\overline{u}) = K - \beta^2, \qquad = K^2\cos^2\theta = Ku_2$$

which shows that $a = 0$.

We now use relations (5.8) to determine the Christoffel symbol and we get

$$K^2\cos^2\theta\,\Gamma^\theta_{\theta\theta} - 2Ku_1\Gamma^\theta_{\theta\varphi} + \frac{u_1^2}{\cos^2\theta}\Gamma^\theta_{\varphi\varphi} = 0,$$
$$-K\partial_\theta u_1 - 2Ku_1\tan\theta + K^2\cos^2\theta\,\Gamma^\varphi_{\theta\theta} - 2Ku_1\Gamma^\varphi_{\theta\varphi} + \frac{u_1^2}{\cos^2\theta}\Gamma^\varphi_{\varphi\varphi} = 0,$$
$$-K^2\cos\theta\,\Gamma^\theta_{\theta\varphi} + \frac{Ku_1}{\cos\theta}\Gamma^\theta_{\varphi\varphi} = 0,$$
$$-K^2\cos\theta\,\Gamma^\varphi_{\theta\varphi} + \frac{Ku_1}{\cos\theta}\Gamma^\varphi_{\varphi\varphi} = K^2\sin\theta,$$

$$\Gamma^\theta_{\varphi\varphi} = \cos\theta(K\sin^2\theta + 2A\sin\theta + K),$$
$$\Gamma^\varphi_{\varphi\varphi} = -\frac{u_1}{K\cos\theta}(K\sin^2\theta + 2A\sin\theta + K)$$

which simplify easily to give

$$\Gamma^\theta_{\theta\varphi} = \frac{u_1}{K\cos\theta}(K\sin^2\theta + 2A\sin\theta + K),$$
$$\Gamma^\varphi_{\theta\varphi} = -\tan\theta - \frac{u_1^2}{K^2\cos^3\theta}(K\sin^2\theta + 2A\sin\theta + K),$$
$$\Gamma^\theta_{\theta\theta} = \frac{u_1^2}{K^2\cos^3\theta}(K\sin^2\theta + 2A\sin\theta + K),$$
$$\Gamma^\varphi_{\theta\theta} = \frac{1}{K\cos^2\theta}\partial_\theta u_1 - \frac{u_1^3}{K^3\cos^5\theta}(K\sin^2\theta + 2A\sin\theta + K).$$

We can now evaluate the size of the tensor field $S = \nabla - \overline{\nabla}$ in terms of the norm given by the metric g_0. We get

$$\begin{aligned} S^2 &= \frac{2u_1^4}{K^4\cos^6\theta}P^2 + \frac{1}{K^2\cos^2\theta}\left[\partial_\theta u_1 - \frac{u_1^3}{K^2\cos^3\theta}P\right]^2 \\ &\quad + \frac{u_1^2}{K^2}P^2 + \frac{1}{\cos^2\theta}[\sin^2\theta + P^2 - 2\sin\theta P], \\ &\geq \frac{1}{\cos^2\theta}(\sin\theta - P)^2, \end{aligned}$$

where $P = K\sin^2\theta + 2A\sin\theta + K$. In order for the connection to extend to the whole of S^2 it is clearly necessary that $\lim_{\theta\to\pm\pi/2} S^2$ remains bounded. Now for this to be true it is necessary that

$$\lim_{\theta\to\pm\pi/2}(\sin\theta - P) = 0$$

i.e.

$$1 - K - 2A - K = 0, \qquad -1 - K + 2A - K = 0$$

which leads to $K = 0$ which is impossible. Hence

Theorem 7.1. *A connection solution of the field equations on the sphere S^2 is necessarily locally symmetric.*

Putting together Theorems 6.1 and 7.1 give:

Theorem 7.2. *A connection solution of the field equations on a compact surface is necessarily locally symmetric.*

8. The symmetric symplectic surfaces

If (M, ω, ∇) is a connected locally symmetric, complete, symplectic manifold, then its universal covering $(\tilde{M}, \tilde{\omega}, \tilde{\nabla})$ is a simply connected symplectic symmetric manifold. Hence $M = \Gamma \setminus \tilde{M}$ where Γ is a group of symplectic affine transformations of $(\tilde{M}, \tilde{\omega}, \tilde{\nabla})$ acting freely and properly discontinuously on $\tilde{M}$.

So the first step is to determine the simply connected symplectic symmetric surfaces. Recall that one associates to a symplectic symmetric space an algebraic object, called a symmetric triple $(\mathcal{G}, \sigma, \Omega)$ [5]. Such a triple is composed of a finite dimensional real Lie algebra $\mathcal{G}$, of an involutive automorphism σ of $\mathcal{G}$ and of a real valued Chevalley 2-cocycle of $\mathcal{G}$, Ω, for the trivial representation of $\mathcal{G}$ on $\mathbb{R}$. These 3 elements satisfy the following axioms:

- If $\mathcal{G} = \mathcal{K}\oplus\mathcal{P}$ where $\sigma_{|\mathcal{K}} = id_{|\mathcal{K}}$ and $\sigma_{|\mathcal{P}} = -id_{|\mathcal{P}}$ then $[\mathcal{P}, \mathcal{P}] = \mathcal{K}$ and the representation $\mathrm{ad}_{|\mathcal{P}}\mathcal{K}$ is faithful.
- For any $k \in \mathcal{K}$, $i(k)\Omega = 0$ and $\Omega_{|\mathcal{P}\times\mathcal{P}}$ is a symplectic form on $\mathcal{P}$.

The dimension of $\mathcal{P}$ is called the dimension of the symmetric triple. It is well known that the isomorphism classes of symmetric triples correspond bijectively to the isomorphism classes of simply connected symplectic symmetric spaces. An easy calculation leads to

Proposition 8.1. *The list of isomorphism classes of two-dimensional symplectic symmetric triples is the following:*

(i) $\mathcal{G} = \mathbb{R}^2$ = *abelian two-dimensional Lie algebra*
 $\sigma = -id_{\mathcal{G}}$
 Ω = *any symplectic form on* $\mathbb{R}^2$

(ii) $\mathcal{G} = sl(2, \mathbb{R})$. *Let* (H, E, F) *be its standard basis. The multiplication reads* $[H, E] = 2E$, $[H, F] = -2F$, $[E, F] = H$.
$\mathcal{K} = \mathbb{R}H$, $\mathcal{P} = \mathbb{R}E \oplus \mathbb{R}F$
$\Omega(E, F) = p \ (\in \mathbb{R}^*)$

(iii) $\mathcal{G} = sl(2, \mathbb{R})$
$\mathcal{K} = \mathbb{R}(E - F)$, $\mathcal{P} = \mathbb{R}H \oplus \mathbb{R}(E + F)$
$\Omega(H, E + F) = p \ (\in \mathbb{R}^*)$

(iv) $\mathcal{G} = so(3, \mathbb{R})$. *Let* (X, Y, Z) *be its standard basis. The multiplication reads* $[X, Y] = Z$, $[Y, Z] = X$, $[Z, X] = Y$.
$\mathcal{K} = \mathbb{R}Z$, $\mathcal{P} = \mathbb{R}X \oplus \mathbb{R}Y$
$\Omega(X, Y) = p \ (\in \mathbb{R}^*)$

(v) $\mathcal{G} = \mathcal{E}(2)$ = *Lie algebra of the isometry group of the euclidian plane. Let* (X, Y, Z) *be its standard basis. The multiplication reads* $[X, Y] = 0$, $[X, Z] = Y$, $[Y, Z] = -X$.
$\mathcal{K} = \mathbb{R}Y$, $\mathcal{P} = \mathbb{R}X + \mathbb{R}Z$
$\Omega(X, Z) = 1$

(vi) $\mathcal{G} = \mathcal{M}(2)$ = *Lie algebra of the isometry group of the minkowski plane. Let* (X, Y, Z) *be its standard basis. The multiplication reads* $[X, Y] = 0$, $[X, Z] = Y$, $[Y, Z] = X$.
$\mathcal{K} = \mathbb{R}X$, $\mathcal{P} = \mathbb{R}Y + \mathbb{R}Z$
$\Omega(Y, Z) = 1$

Remarks.

(i) *In case II, the subspace* $\mathbb{R}E$ *(resp.* $\mathbb{R}F$*) is stabilized by ad* $\mathcal{K}$. *In case V, the subspace* $\mathbb{R}X$ *is stabilized by ad* $\mathcal{K}$. *In case VI, the subspace* $\mathbb{R}Y$ *is stabilized by ad* $\mathcal{K}$.

(ii) *In cases II, III, IV, the isomorphism class is determined by a parameter* p, *which measures the "size" of the symplectic form.*

(iii) *In case II, the symmetric bilinear form* Q *on* $\mathcal{P}$ *defined by*

$$Q(E, E) = Q(F, F) = 0, \qquad Q(E, F) = 1$$

is ad $\mathcal{K}$ *invariant and of signature* $(1, 1)$.

In case III, the symmetric bilinear form Q *on* $\mathcal{P}$ *defined by*

$$Q(H, H) = 1 = Q(E + F, E + F), \qquad Q(H, E + F) = 0$$

is ad $\mathcal{K}$ *invariant and of signature* $(2, 0)$.

In case IV, the symmetric bilinear form Q *on* $\mathcal{P}$ *defined by*

$$Q(X, X) = 1 = Q(Y, Y), \qquad Q(X, Y) = 0$$

is ad $\mathcal{K}$ *invariant and of signature* $(2, 0)$.

(iv) *In cases V, VI, there are no non-degenerate ad* $\mathcal{K}$ *invariant symmetric bilinear form on* $\mathcal{P}$.

The description of the simply connected symplectic symmetric surfaces corresponding to the list of Proposition 1 is as follows:

(i) The plane $\mathbb{R}^2$ with its standard symplectic structure ω_0, and the standard flat affine connection.

(ii) Consider the adjoint orbit of the element H of $sl(2,\mathbb{R})$. If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $(ad-bc=1)$ denotes an element of $SL(2,\mathbb{R})$, a point of the orbit of H has components

$$(ad+bc)H - 2abE + 2cdF.$$

It is a one-sheeted hyperboloïd in $sl(2,\mathbb{R})$. The symplectic form is the standard Kostant Souriau form. Its symmetric symplectic connection is the Levi Civita connection associated to the Lorentz metric, which is the constant negative scalar curvature Lorentz metric. The model for II is the universal covering of this orbit.

(iii) Consider the adjoint orbit of the element $E-F$ of $sl(2,\mathbb{R})$. With the same notations as above, a point of this orbit has components

$$-(bd+ac)H + (b^2+a^2)E - (c^2+d^2)F.$$

It is one connected component of a two-sheeted hyperboloïd in $sl(2,\mathbb{R})$. The symplectic form is the standard Kostant Souriau form. The symmetric symplectic connection is the Levi Civita connection associated to the Riemann metric, which is of constant negative scalar curvature. It is diffeomorphic to $\mathbb{R}^2$.

(iv) Consider the adjoint orbit of the element Z in $so(3,\mathbb{R})$. It is the standard 2-sphere with its canonical symplectic structure. The symmetric symplectic connection is the Levi Civita connection associated to the standard metric of S^2.

(v) Consider the coadjoint orbit of the element Y^* of $\mathcal{E}(2)^*$ (where X^*, Y^*, Z^* is the dual basis of the basis of $\mathcal{E}(2)$ given in Proposition 1). If one denotes by (θ, a, b) an element of $E(2)$, a point of the orbit of Y^* has components

$$(a\cos\theta + b\sin\theta)Z^* - \sin\theta X^* + \cos\theta Y^*.$$

It is a circular cylinder. The model for V is the universal covering of this orbit.

(vi) Let $\mathcal{M}(2)$ be the connected component of the group of isometries of the Lorentz plane $\mathbb{R}^2$, $g_0 = 2da\otimes db$. Let (t,u,v) be an element of $M(2)$ acting by

$$(t,u,v)\cdot(a,b) = (\mathrm{e}^t a + u, \mathrm{e}^{-t} b + v).$$

Let (Z, X, Y) be the basis of the Lie algebra $\mathcal{M}(2)$ given by

$$Z = \partial_t|_0, \qquad X = \partial_u|_0, \qquad Y = \partial_v|_0,$$

and let Z^*, X^*, Y^* be the dual basis of $\mathcal{M}(2)^*$. A point in the coadjoint orbit of the element (X^*+Y^*) has component

$$(-u\mathrm{e}^{-t} - v\mathrm{e}^t)Z^* + \mathrm{e}^{-t}X^* + \mathrm{e}^t Y^*$$

it is a connected component of an hyperbolic cylinder in $\mathcal{M}(2)^*$ and is thus diffeomorphic to $\mathbb{R}^2$.

9. The two-dimensional compact complete locally symmetric symplectic space

Any such space (M, ω) is a quotient of its universal cover $(\tilde{M}, \tilde{\omega})$ (which is symmetric symplectic surface) by a group Γ of symplectic affine transformations acting freely and properly discontinuously on $(\tilde{M}, \tilde{\omega})$. So we examine successively the cases given in Proposition 8.1.

For case I, the plane $\mathbb{R}^2$ with its standard symplectic form and with a flat torsion-free affine connection, we are helped by the following result of Milnor [11].

Proposition 9.1. *A compact orientable surface of genus $g \geq 2$ does not admit an affine connection with zero curvature. Hence we have*

Lemma 9.1. *Any compact symplectic surface having the standard symplectic plane $(\mathbb{R}^2, \omega_c)$ with a flat torsion-free affine connection as universal cover is necessarily a flat torus.*

For case II, the universal cover $(\tilde{M}, \tilde{\omega})$ of the adjoint orbit of H in $\mathrm{sl}(2, \mathbb{R})$, we first describe the automorphism group of $(\tilde{M}, \tilde{\omega})$. It is known [5–7] that the automorphism group A is the intersection of the affine group of $\tilde{M}$ with the group of symplectic diffeomorphisms of $(\tilde{M}, \tilde{\omega})$. It is also known [6,7,12] that the algebra $\mathcal{A}$ of A contains the algebra $\mathcal{G}$ of the transvection group G of $\tilde{M}$ and is composed of derivations of this algebra.

As $\mathcal{G} = \mathrm{sl}(2, \mathbb{R})$, $\mathcal{A} = \mathcal{G}$ and the transvection group G is the identity component of A. Let a be any automorphism of $\tilde{M}$ stabilizing a base point $\tilde{o}$ $(\in \tilde{M})$; the differential $a_{*\tilde{o}}$ being symplectic, belongs to $\mathrm{SL}(2, \mathbb{R})$. It is also affine and thus commutes with the action of the curvature endomorphism. Taking a basis E, F of $\tilde{M}_{\tilde{o}}$, as in Proposition 8.1 and with obvious identifications, one sees that

$$(a_{*_{\tilde{o}}}) = \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \qquad r \in \mathbb{R}_0.$$

Composing a_{*_0} with the differential of an element k of the stabilizer of $\tilde{o}$ in G, one can reduce $(a_{*_{\tilde{o}}})$ to $\in I$. The symmetry $s_{\tilde{o}}$ at $\tilde{o}$ has differential $-I$ and thus coincides with a. As $s_{\tilde{o}}$ is an isometry of $\tilde{M}$, for its Lorentz metric, we conclude that the automorphism group of $\tilde{M}$ is composed of isometries and hence that any affine compact quotient admits a Lorentz metric of constant negative scalar curvature.

It is well known [13] that any compact manifold admitting a Lorentz metric has vanishing Euler–Poincaré characteristic; hence in dimension 2 is a torus T^2.

One also known [14] that the Gauss–Bonnet theorem is valid independently of the signature; hence that

$$\chi = \frac{1}{\pi} \int \tau v_g,$$

where τ is the scalar curvature of the Lorentz metric.

This implies in particular that there cannot exist a constant, non-zero curvature Lorentz metric on T^2. Hence we have

Lemma 9.2. *The universal cover of the orbit of H in $sl(2, \mathbb{R})$ does not admit a compact, affine symplectic quotient.*

For case III, the adjoint orbit of E–F is $sl(2, \mathbb{R})$, we first determine (as for case II) the automorphism group A of the orbit. The algebra $\mathcal{A}$ of A coincides with the algebra $\mathcal{G}$ of the transvection group G as $\mathcal{G} = sl(2, \mathbb{R})$. If a is any automorphism of the orbit, stabilizing $(E - H)$, its differential $a_{*_{E-F}}$ being symplectic belongs to $SL(2, \mathbb{R})$. As it commutes with the curvature endomorphism, it has, relative to the $(H, E + F)$ basis of the tangent space to the orbit at $E - F$, the matrix:

$$(a_{*_{E-F}}) = \begin{pmatrix} r & -s \\ s & r \end{pmatrix}, \quad r^2 + s^2 = 1.$$

Hence $a_{*_{E-F}}$ coincides with the differential of an element k of the stabilizer of $E - F$ in G. Hence $A = G$ and the automorphism group of the orbit is composed of isometries of the orbit relative to its Riemannian metric of constant negative curvature. In fact all these isometries are holomorphic with respect to the standard almost complex structure on the orbit. Hence the group Γ of such transformations of the covering is a group of holomorphic isometries. The sought for compact quotient is thus necessarily a compact Riemann surface endowed with a metric of constant negative curvature. Using Gauss–Bonnet one has

Lemma 9.3. *The orbit of $E - F$ in $sl(2, \mathbb{R})$ admits as possible compact quotients an orientable surface of genus $g \geq 2$.*

Remark. *Poincaré's theorem [9] ensures the existence, on any orientable compact surface of genus $g \geq 2$ of a Riemannian metric of constant negative curvature.*

For case IV, the orbit of Z in $so(3, \mathbb{R})$, one shows in a completely analogous way that the automorphism group of the orbit (the standard sphere S^2) is the three-dimensional rotation group $SO(3)$. One checks easily that the rotation group does not admit a discrete subgroup acting freely on S^2.

Lemma 9.4. *The sphere S^2 does not admit a non-trivial affine symplectic quotient.*

For case V, the universal cover $(\tilde{M}, \tilde{\omega})$ of the orbit of Y^* in $\mathcal{E}(2)^*$, (we use the notations of Proposition 8.1) we start as above to determine the automorphism group A of $(\tilde{M}, \tilde{\omega})$. If a is an element of A stabilizing the point base $\tilde{o}(\in \tilde{M})$, $a_{*_{\tilde{o}}} \in SL(2, \mathbb{R})$; furthermore as it commutes with the curvature endomorphism, its matrix , in the basis $\{X, Z\}$ given in Proposition 9.1, is:

$$(a_{*_{\tilde{o}}}) = \begin{pmatrix} \epsilon & c \\ 0 & \epsilon \end{pmatrix} \quad \epsilon^2 = 1,\ c \in \mathbb{R}.$$

If $k \in \mathcal{K}$, an element $\exp k$ of the isotropy subgroup of the transvection group G of $\tilde{M}$ has a differential of the form:

$$(\exp k)_{*\tilde{o}} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}.$$

We can assume that $a_{*\tilde{0}} = \pm I$ and clearly $-I$ is the differential at $\tilde{0}$ of $s_{\tilde{0}}$, which is an automorphism of $\tilde{M}$.

Hence A has two connected components, G being the identity component of A.

We now show that the universal cover of the group $E(2)$ is the transvection group of $\tilde{M}$ ($\sim \mathbb{R}^2$) and describe its action on $\mathbb{R}^2$.

The universal cover of $E(2)(\sim \mathbb{R}^2)$ is the set of pairs $\{(\theta, \alpha) | \theta \in \mathbb{R},\ \alpha \in \mathbb{C}\}$ with the multiplication

$$(\theta, \alpha)(\theta_1, \alpha_1) = (\theta + \theta_1,\ e^{i\theta}\alpha_1 + \alpha).$$

The involutive automorphism of this group corresponding to our algebraic model is

$$\sigma(\theta, \alpha) = (-\theta, -\bar{\alpha}).$$

The group of fixed points of σ is $K = \{(0, ir) | r \in \mathbb{R}\}$. The projection $\pi : G \to G/K \sim \tilde{M}$ is given by

$$\pi(\theta, \alpha) = (\theta, u = \cos\theta \operatorname{Re}\alpha + \sin\theta \operatorname{Im}\alpha).$$

The action of G on $\tilde{M}$ is:

$$(\theta, \alpha), (\theta_1, u_1) = (\theta + \theta_1, u_1 + \cos(\theta + \theta_1)\operatorname{Re}\alpha + \sin(\theta + \theta_1)\operatorname{Im}\alpha).$$

The action is effective and thus G is the transvection group of $\tilde{M}$.

The curvature endomorphism stabilizes the direction of X. As the cover $\tilde{M} \to \tilde{M}/\Gamma = M$ is affine, there exists on $\tilde{M}/\Gamma$ a smooth field of directions. As M is compact we have $\chi(M) = 0$ and hence M (if it exists) must be diffeomorphic to the torus T^2. The group Γ is thus isomorphic to $\mathbb{Z}^2$ and we are thus going to describe, up to conjugation, the discrete subgroups of A isomorphic to $\mathbb{Z}^2$, and their action on $\tilde{M}$.

As A has two connected components, $\Gamma \cap A_0$ is a subgroup Γ_1 of finite index in Γ, isomorphic to $\mathbb{Z}^2$. We shall only describe, up to conjugation the subgroups Γ_1 of G. Now if $(a, z) \in G$, its conjugacy class is composed of all elements

$$(\theta, \alpha)(a, z)(\theta, \alpha)^{-1} = (a, (1 - e^{ia})\alpha + e^{i\theta}z) \quad \theta \in \mathbb{R},\ \alpha \in \mathbb{C}.$$

We have thus two types of conjugacy classes

(1) $a = 2k\pi \Rightarrow \exists$ a representative of the class where $\operatorname{Im} z = 0$, $\operatorname{Re} z \geq 0$.

(2) $a \neq 2k\pi \Rightarrow \exists$ a representative of the class where $z = 0$.

Let γ_1 be one of the generators of Γ_1 and assume that the conjugacy class of γ_1 is of type (1). We may thus assume

$$\gamma_1 = (2k\pi, r), \qquad r \in \mathbb{R}_0^+.$$

If γ_2 is the other generator of Γ_1, the fact that γ_1 and γ_2 commute implies:

$$\begin{aligned} &\gamma_2 = (2l\pi, \alpha) && \text{if } r > 0, \\ &\text{no condition on } \gamma_2 && \text{if } r = 0. \end{aligned}$$

In this second part of the alternative, we can by a conjugation stabilizing γ_1 assume that γ_2 has one of the two following forms

$$\gamma_2 = (2m\pi, s), \qquad s \geq 0, \tag{a}$$

or

$$\gamma_2 = (\theta, 0), \qquad \theta \neq 2l\pi. \tag{b}$$

If γ_2 is of type (a) we have for the action of the generators on $\tilde{M}$:

$$\gamma_1(x, u) = (x + 2k\pi, u), \qquad \gamma_2(x, u) = (x + 2l\pi, u + \cos xs).$$

This shows that $\gamma_2^k \gamma_1^{-l}$ fixes all points of the form $((2p+1)\frac{\pi}{2}, u)$ and thus the action of Γ_1 is not free.

If γ_2 is of type (b), the action of the generators on $\tilde{M}$ is given by:

$$\gamma_1(x, u) = (x + 2k\pi, u), \qquad \gamma_2(x, u) = (x + \theta, u)$$

and shows that $\tilde{M}/\Gamma_1$ cannot be compact.

In the first part of the alternative we have:

$$\begin{aligned} \gamma_1(x, u) &= (x + 2k\pi, u + \cos xr), \\ \gamma_2(x, u) &= (x + 2l\pi, u + \cos x \operatorname{Re}\alpha + \sin x \operatorname{Im}\alpha), \end{aligned}$$

and thus

$$\gamma_2^m \gamma_1^n(x, u) = (x + 2(nk + lm)\pi, u + \cos x(nr + m\operatorname{Re}\alpha) + \sin x(m\Im\alpha)).$$

One can choose m and n such that $nk + lm = 0$ and one can choose x such that $\cos x(nr + m\operatorname{Re}\alpha) + \sin x(m\Im\alpha) = 0$. Hence there are fixed points and the action of Γ_1 is not free.

The second case, where $\gamma_1 = (a, 0)$ $(a \neq 2l\pi)$ can be treated similarly. The second generator $\gamma_2 = (\theta, \alpha)$ commuting with γ_1 must be of the form $(\theta, 0)$ and one checks as above that the quotient of $\tilde{M}$ by Γ_1 cannot be compact. We conclude by

Lemma 9.5. *The universal cover of the orbit of Y^* in $\mathcal{E}(2)^*$ has no affine symplectic compact quotient.*

In case VI, the coadjoint orbit (M, ω) of X^* in $\mathcal{M}(2)^*$, we proceed essentially as in case V.

Let $a \in A(=$ automorphism group of $M)$ and assume a stabilizes a basis point o $(\in M)$. Then $a_{*0} \in SL(2, \mathbb{R})$ and commutes with the curvature endomorphism. Hence in the basis $\{Y, Z\}$ of M_0 (cf. Proposition 8.1) it has a matrix of the form

$$(a_{*0}) = \begin{pmatrix} \epsilon & 0 \\ c & \epsilon \end{pmatrix}, \quad \epsilon^2 = 1, \ c \in \mathbb{R}.$$

As an element of the form $\exp k$ ($k \in \mathcal{K}$) has a differential in 0 of the form:

$$(\exp k)_{*0} = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}$$

and as $-I$ coincides with the differential of the symmetry at 0, we see that A has two connected components and that A_0 = identity component of the isometry group of the Minkowski plane = G = transvection group of M.

As the curvature endomorphism stabilizes one direction, any compact, affine symplectic quotient V of M will have $\chi(V) = 0$ and hence V is diffeomorphic to T^2.

So $V = M/\Gamma$, where Γ is a subgroup of A isomorphic to $\mathbb{Z}^2$. Now, as $\Gamma_1 = \Gamma \cap G$ has finite index in Γ, it is still isomorphic to $\mathbb{Z}^2$ and it will be sufficient, for our purpose, to investigate the quotients M/Γ_1. To classify, up to conjugation, the subgroups of G, isomorphic to $\mathbb{Z}^2$ acting freely and properly discontinuously on M we now describe more explicitly G and M.

Let $G = \{(t, u, v) \in \mathbb{R}^3\}$ with the multiplication given by

$$(t, u, v)(t', u', v') = (t + t', \mathrm{e}^t u' + u, \mathrm{e}^{-t} v' + v).$$

and let σ be the involution automorphism defined by

$$\sigma(t, u, v) = (-t, v, u).$$

Then the group K of fixed points of σ is

$$K = \{(0, s, s) | s \in \mathbb{R}\}.$$

The affine symmetric space G/K is precisely our orbit. Indeed if we define:

$$Z = \frac{\mathrm{d}}{\mathrm{d}t}(t, 0, 0)_{|0}, \qquad X = \frac{\mathrm{d}}{\mathrm{d}t}(0, t, t)_{|0}, \qquad Y = \frac{\mathrm{d}}{\mathrm{d}t}(0, -t, t)_{|0}.$$

One checks that the commutation relations coincide with the ones given in Proposition 8.1.

Let $\pi : G \to G/K = M : (t, u, v) \to (t, w = \mathrm{e}^{-t} u - \mathrm{e}^t v)$ be the canonical projection. The action of G on M is given by

$$(t, u, v) \cdot (s, w) = (s + t, w + u\mathrm{e}^{-(s+t)} - v\mathrm{e}^{(s+t)}).$$

Now, if $(a, b, c) \in G$, its conjugacy class is the set of elements of the form

$$\begin{aligned}&(t, u, v)(a, b, c)(t, u, v)^{-1} \\ &\quad = (a, \mathrm{e}^t b + (1 - \mathrm{e}^a)u, \mathrm{e}^{-t} c + (1 - \mathrm{e}^{-a})v), \quad t, u, v \in \mathbb{R}.\end{aligned}$$

So we have two types of conjugacy classes (not counting the trivial one)

(a) the conjugacy class contains an element of the form

$$(0, \epsilon, c), \qquad \epsilon^2 = 1,$$

(b) the conjugacy class contains an element of the form

$$(a, 0, 0), \qquad a \neq 0.$$

One checks that elements of type (a) never commute with elements of type (b). Hence if γ_1, γ_2 are generators of Γ_1 one may assume either
(1) γ_1, γ_2 of type (a) or
(2) γ_1, γ_2 of type (b)
In case (1) we have

$$\gamma_1 = (0, \epsilon, c), \qquad \gamma_2 = (0, b', c')$$

and

$$\gamma_1(s, w) = (s, w + \epsilon \mathrm{e}^{-s} - c\mathrm{e}^{s}), \qquad \gamma_2(s, w) = (s, w + b'\mathrm{e}^{-s} - c'\mathrm{e}^{s})$$

Thus the quotient by Γ_1 cannot be compact.
In case (2), we have

$$\gamma_1 = (a, 0, 0), \qquad \gamma_2 = (a', 0, 0)$$

and

$$\gamma_1(s, w) = (s + a, w), \qquad \gamma_2(s, w) = (s' + a, w).$$

Thus the quotient by Γ_1 cannot be compact. We conclude by:

Lemma 9.6. *The orbit of X^* in $\mathcal{M}(2)^*$ has no affine symplectic compact quotient.*

Summarizing the Lemmas 1 – 6 we get

Theorem 9.1. *Let (M, ω, ∇) be a compact locally symmetric symplectic, complete, two-dimensional manifold. Then either*
- *M is the sphere S^2 with a multiple of its standard symplectic structure ω_0 and ∇ is the Levi Civita connection associated to its standard Riemannian structure g_0 (with constant positive curvature),*
- *M is the torus T^2 with a flat affine connection,*
- *M is a surface of genus $g (g \geq 2)$, Σ_g, with a connection ∇ which is the Levi Civita connection associated to a metric h of constant negative curvature.*

10. The two-dimensional case: a global approach II

We have proven (Theorem 7.3) that if a compact symplectic surface (M, ω) admits a symplectic connection ∇ solution of our field equations, then (M, ω, ∇) is a locally symmetric symplectic surface. We have also proven (Theorem 9.1) that a complete locally symmetric symplectic compact surface (M, ω, ∇) necessarily belongs to a short list. We would like to conclude by determining for any compact symplectic surface (M, ω) the set of symplectic connections ∇ which are solutions of our field equations and geodesically complete. Clearly the symplectic diffeomorphism group $S(M, \omega)$ acts on this set of connections and what we are really interested in is to describe the set $\mathcal{E}$ of equivalence classes of such connection modulo the action of $S(M, \omega)$. We shall proceed case by case.

Consider any volume form ω on S^2. Let on the other hand g_0 be the Riemannian metric on the sphere, with constant curvature $= +1$ and let ω_0 be the associated Riemannian volume form chosen in such a way that ω_0 and ω belong to the same orientation. (S^2, g_0) is a Riemannian symmetric space and up to homothety is uniquely determined by S^2. The Levi Civita connection ∇^0 of g_0 is a symplectic connection for ω_0, is complete and is a solution of our field equations for (S^2, ω_0) and in fact the only one, up to isometry [5–7]. There exists a positive real number k such that

$$\int_{S^2} \omega = \int_{S^2} k\omega_0.$$

This says that the corresponding de Rham cohomology classes are equal:

$$[\omega] = [k\omega_0].$$

Thus Moser's stability theorem [8] tells us that there exists a smooth isotopy φ_t $(0 \leq t \leq 1)$ such that: (i) $\varphi_0 = \mathrm{id}_{|_{S^2}}$; (ii) $\forall t,\ \varphi_t$ is a diffeomorphism of S^2; (iii) $\varphi_1^*\omega = k\omega_0$. Clearly if g is the Riemannian metric defined by

$$\varphi_1^* g = g_0$$

and if ∇ is the Levi Civita connection associated to g, ∇ is a symplectic connection for ω, is complete and is a solution of our field equations. Hence we have existence. Furthermore if ∇' is another symplectic connection, solution of the field equations relative to ω, we know that it is symmetric (as S^2 is simply connected) and thus can be obtained from ∇^0 as ∇ and hence differs from ∇ by an element of $S(S^2, \omega)$.

Lemma 10.1. *Let ω be a volume form on S^2. Then there exists a symplectic connection ∇ which is complete and solution of our field equations; furthermore two such connections are related by a symplectic diffeomorphism. The set $\mathcal{E}$ consists of only one point.*

Consider any volume form on the torus T^2. Let on the other hand ω_0 be the "constant" volume form on T^2 such that it belongs to the orientation given by ω and such that

$$\int_{T^2} \omega = \int_{T^2} \omega_0.$$

As above there exists a smooth isotopy φ_t such that

$$\varphi_1^*\omega = \omega_0.$$

If $\pi : \mathbb{R}^2 \to T^2$ denotes the standard covering, $\pi^*\omega_0$ is a constant symplectic form on $\mathbb{R}^2$. The symplectic vector space $(\mathbb{R}^2, \pi^*\omega_0)$ admits a flat complete symplectic affine connection $\tilde{\nabla}_0$ such that $(\mathbb{R}^2, \pi^*\omega_0, \tilde{\nabla}_0)$ is an affine symmetric space. This connection is unique up to a symplectic affine transformation of $(\mathbb{R}^2, \pi^*\omega_0)$. The covering π is induced by the action of a free abelian subgroup Γ of $\mathbb{R}^2$ with two generators acting properly discontinuously and freely on $\mathbb{R}^2$. Clearly the elements of Γ are affine symplectic transformations of $(\mathbb{R}^2, \pi^*\omega_0, \tilde{\nabla}_0)$ and thus $\tilde{\nabla}_0$ induces a flat affine symplectic connection ∇_0 on (T^2, ω_0). This is locally symmetric complete and solution of our field equations relative to ω_0. Hence the connection ∇ defined by

$$\nabla_X Y = \varphi_{1*} \nabla^0_{\varphi_{1*}^{-1} X} \varphi_{1*}^{-1} Y$$

is a solution of our field equations for ω. Hence we have existence.

Conversely if (T^2, ω, ∇_1) and (T^2, ω, ∇_2) are two solutions of our field equations, they are locally symmetric, flat, complete, symplectic. Hence they are both covered by the standard symmetric symplectic vector space $(\mathbb{R}^2, \bar{\omega}, \nabla)$ and are obtained by the action of a discrete subgroup Γ_1 (resp. Γ_2) of the affine symplectic group A of $(\mathbb{R}^2, \bar{\omega}, \nabla)$. Now $A = SL(2, \mathbb{R}) \cdot \mathbb{R}^2$ and Γ_1 (resp. Γ_2) is isomorphic to $\mathbb{Z}^2$ and acts freely and properly discontinuously on $\mathbb{R}$.

The torus (T^2, ω, ∇_1) and (T^2, ω, ∇_2) are affinely, symplectically equivalent iff the groups Γ_i are conjugate in A.

Lemma 10.2. *Let ω be a volume form on T^2. Then there exists a symplectic connection ∇ which is complete and solution of our field equations. Furthermore the set $\mathcal{E}$ of equivalence classes of such solutions is the set of conjugacy classes of discrete subgroups Γ of $SL(2, \mathbb{R}) \cdot \mathbb{R}^2$, isomorphic to $\mathbb{Z}^2$ and acting freely and properly discontinuously on $\mathbb{R}^2$*

Consider any volume form ω on the surface Σ_g of genus $g \geq 2$. By Poincaré's theorem we know that there exists on Σ_g a Riemannian metric of negative scalar curvature $= -1$. The space (Σ_g, g_0) is locally symmetric and complete, let ω_0 be the Riemannian volume form belonging to the orientation defined by ω. The Levi Civita connection ∇_0 of g_0 is a symplectic connection solution of our field equations relative to ω_0. There exists a positive real number k such that

$$\int_{\Sigma_g} \omega = \int_{\Sigma_g} k\omega_0.$$

Using the same argument as above we get existence of a solution of the field equations for ω. Let $(\Sigma_g, \omega, \nabla_1)$ $(i = 1, 2)$ be a solution of our field equations, which is complete. Then it is locally symmetric and complete. Hence it is affinely, symplectically covered by the unit disk (equivalently by the orbit of $E - F$ in $sl(2, \mathbb{R})$) with its standard symmetric structure, its standard Riemannian metric of constant negative curvature -1 and with symplectic form a constant multiple of the standard symplectic form. In the proof of Lemma 9.3, we have shown that the group Γ of deck transformation of the covering $\pi : D \to \Sigma_g$ is composed of holomorphic isometries of D. Hence (Σ_g, ω) inherits a Riemannian metric of constant negative curvature equals to -1. Observe also that if we are given h, the orienation of Σ_g and the real number k, ω is uniquely determined. So if there exists an affine symplectomorphism φ of $(\Sigma_g, \omega, \nabla_1) \to (\Sigma_g, \omega, \nabla_2)$ this is also a holomorphic isometry for the corresponding metrics h_1 and h_2. Conversely such a holomorphic isometry induces an affine equivalence of our two solutions. Hence we have

Lemma 10.3. *Let ω be a volume form on a compact surface Σ_g of genus $g \geq 2$. Then there exists a symplectic connection ∇ which is complete and solution of our field equations. Furthermore the set $\mathcal{E}$ of equivalence classes of such solutions is isomorphic to the set of equivalence classes of constant curvature $(= -1)$ Riemannian metrics on Σ_g modulo diffeomorphisms which stabilize ω.*

Putting together Lemmas 10.1–10.3, we get

Theorem 10.1. *Let ω be a volume form on a compact surface M. Then*
(i) *If $M = S^2$, $\mathcal{E} = 1$ point*
(ii) *If $M = T^2$, $\mathcal{E}$ = set of conjugacy classes of discrete subgroups Γ of $SL(2, \mathbb{R}) \cdot \mathbb{R}^2$ isomorphic to $\mathbb{Z}^2$ and acting freely and properly discontinuously on $\mathbb{R}^2$*
(iii) *If $M = \Sigma_g (g \geq 2)$ then $\mathcal{E}$ = set of equivalence classes of constant curvature $(= -1)$ metrics modulo the group of diffeomorphisms stabilizing ω.*

Remarks. *Clearly if ϕ is a diffeomorphism of Σ_g, it sends a solution for ω on a solution for $\phi^*\omega$. Hence if one is interested in the space of equivalence classes of solutions on Σ_g one has either the classical moduli space or Teichmüller space.*

11. The two dimensional case: a local approach II

We now want to investigate the non-compact situation. From the start we will restrict ourselves to the case of $\mathbb{R}^2$, and we shall also assume that the preferred connection we want to describe is geodesically complete. From our previous analysis we get

Proposition 11.1. *Let $(\mathbb{R}^2, \omega, \nabla)$ be a preferred symplectic connection on the plane which is locally symmetric and complete. Then it is globally symmetric and the manifold is symplectomorphic and affinely equivalent to one of the following manifolds:*
(i) *$(\mathbb{R}^2, \omega_0, \nabla_0)$ viewed as a symplectic vector space and ∇ is the flat affine connection.*
(ii) *$(\mathbb{R}^2, \omega_1, \nabla_1)$ viewed as the universal cover of the coadjoint orbit of the element H of $sl(2, \mathbb{R})$. The symplectic form is a multiple of the Kostant Souriau form. The connection is the Levi Civita connection associated to the Lorentz metric on this orbit which is induced by the Killing form of $sl(2, \mathbb{R})$.*
(iii) *$(\mathbb{R}^2, \omega_2, \nabla_2)$ viewed as the coadjoint orbit of the element $(E - F)$ of $sl(2, \mathbb{R})$. The symplectic form is a multiple of the Kostant Souriau form. The connection is the Levi*

Civita connection associated to the Riemannian metric on this orbit which is induced by the Killing form of $sl(2, \mathbb{R})$.

(iv) $(\mathbb{R}^2, \omega_3, \nabla_3)$ *viewed as the universal cover of the coadjoint orbit of the element* Y^* *of* $\mathcal{E}(2)^*$. *The symplectic form is the Kostant Souriau form and the connection is the unique linear symplectic connection for which the symmetries are affine transformations.*

(v) $(\mathbb{R}^2, \omega_4, \nabla_4)$ *viewed as the coadjoint orbit of the element* $X^* + Y^*$ *of* $\mathcal{M}(2)^*$. *The symplectic form is the Kostant Souriau form and the connection is the unique linear symplectic connection for which the symmetries are affine transformations.*

Remarks. *This proposition indicates how complicated the non-compact situation is. Indeed all these spaces are symplectomorphic to the standard symplectic vector space (i.e. $\exists$ a global Darboux chart) but no two of these preferred connections are affinely equivalent.*

We recall that the local behavior of a solution of the field equations is governed by a function β. The Proposition 11.1 *takes care of the case* $\beta = 0$.

Proposition 11.2. *Let* $(\mathbb{R}^2, \omega, \nabla)$ *be a solution of the field equations. Assume the function* β *is constant* ($\beta = \beta_0 \neq 0$). *Then if* $U = \{p \in \mathbb{R}^2 \mid u(p) \neq 0\}$, U *is an open dense subset of* $\mathbb{R}^2$. *Furthermore if* $p \in U$, *there exists a Darboux chart* (V, φ) *centered at* p, $(V \subset U)$, *with coordinates* x^a $(a = 1, 2)$ *such that*

(i) $\omega|_V = \mathrm{d}x^1 \wedge \mathrm{d}x^2$

(ii) $u|_V = -2\beta_0(1 + x^1)\,\mathrm{d}x^2$

(iii) *the Christoffel symbols of* ∇ *on* V *have the expression*

$$\nabla_{\partial_1}\partial_1 = -\frac{1}{2(1+x^1)}\partial_1, \qquad \nabla_{\partial_1}\partial_2 = \frac{1}{2(1+x^1)}\partial_2,$$
$$\nabla_{\partial_2}\partial_1 = \nabla_{\partial_1}\partial_2, \qquad \nabla_{\partial_2}\partial_2 = -2\beta_0(1+x^1)^3\partial_1.$$

Proof. The assumption $\beta = \beta_0 \neq 0$ implies that the field equations reduce to

$$r_{ab} = \frac{1}{\beta_0}u_a u_b, \qquad u_{a;b} = \beta_0\omega_{ab}.$$

The set $U = \{p \in \mathbb{R}^2 \mid u(p) = 0\}$ is open. Furthermore if $q \in \mathbb{R}^2 \setminus U$ is an interior point of $\mathbb{R}^2 \setminus U$, we have a contradiction as on one hand $\mathrm{d}u|_q = 0$ and on the other hand $\mathrm{d}u|_q = -2\beta_0\omega \neq 0$. Hence the density of U. Let $p \in U$ and let $\Omega_p \subset U$ be the domain of a chart with coordinates x^a $(a = 1, 2)$ such that

(i) $x^a(p) = 0$

(ii) if $\bar{u}$ is defined by $i(\bar{u})\omega = u$, then $\bar{u} = \partial/\partial x^1$

In this chart

$$\omega = a(x)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2, \qquad u = a(x)\,\mathrm{d}x^2.$$

Recall that

$$\mathrm{d}u = -2\beta_0\omega \neq 0 = \mathrm{d}a \wedge \mathrm{d}x^2$$

Hence $\mathrm{d}a$ and $\mathrm{d}x^2$ are linearly independent and there exists $\Omega_p^1 \subset \Omega_p$ such that a and x^2 form a coordinate system on Ω_p^1. Write $\tilde{x}^a$ for the coordinates on Ω_p^1 such that

$$a = -2\beta_0(1+\tilde{x}^1), \qquad \tilde{x}^2 = x^2.$$

Thus

$$u = -2\beta_0(1+\tilde{x}^1)\,\mathrm{d}\tilde{x}^2, \qquad \omega = \mathrm{d}\tilde{x}^1 \wedge \mathrm{d}\tilde{x}^2.$$

Now on Ω_p^1 the relations $u_{a;b} = \beta_0\omega_{ab}$ and the fact that the connection is symplectic give the value of 5 of the 6 Christoffel symbols:

$$\nabla_{\partial_1}\partial_1 = -\frac{1}{2(1+x^1)}\partial_1, \qquad \nabla_{\partial_1}\partial_2 = \nabla_{\partial_2}\partial_1 = \frac{1}{2(1+x^1)}\partial_2,$$
$$\nabla_{\partial_2}\partial_2 = A(x^1, x^2)\partial_1.$$

where A is an arbitrary function (we have omitted the $\tilde{}$ on the x^a's). The components of the Ricci tensor are given in terms of the components of u:

$$r_{11} = r_{12} = 0, \qquad r_{22} = 4\beta_0(1+x^1)^2.$$

On the other hand, computing the Ricci tensor in terms of the Christoffel symbols and their derivatives, we get the condition

$$-\partial_1 A + \frac{A}{1+x^1} = 4\beta_0(1+x^1)^2. \tag{11.1}$$

The most general solution of (11.1) is given by

$$A = -2\beta_0(1+x^1)^3 + B(x^2)(1+x^1)$$

and depends on an arbitrary function B of one variable.

We are now going to use the arbitrariness of the coordinates on Ω_p^1 to get rid of the function B on an eventually smaller neighborhood Ω_p^2.

A change of local coordinates compatible with the fact that the chart is Darboux and the particular form of u, is necessarily of the form:

$$x^1 = \frac{1}{\dot{\varphi}}(1+x'^1) - 1, \qquad x^2 = \varphi(x'^2).$$

where $\dot{\varphi} = d\varphi/dx'^2$. In this coordinate system, the Christoffel symbols have the same expression except that the function $B(x^2)$ is replaced by the function

$$B'(x'^2) = \dot{\varphi}^2(B\circ\varphi) + \frac{\ddot{\varphi}^2}{2\dot{\varphi}^2} - \left(\frac{\ddot{\varphi}}{\dot{\varphi}}\right)^{\cdot}.$$

So we are looking for a function φ such that

$$2\dddot{\varphi}\dot{\varphi} - 3\ddot{\varphi}^2 = 2\dot{\varphi}^4(B\circ\varphi).$$

There exists a solution φ such that $\varphi(0) = 0$, $\dot{\varphi}(0) = 1$, $\ddot{\varphi}(0) = 0$ defined on a small interval centered at 0. This allows us to define the neighborhood $V = \Omega_p^2$. □

Proposition 11.3. *Let $(\mathbb{R}^2, \omega_0)$ be the standard symplectic vector space with coordinates $x^a (a = 1, 2)$. Let $U^\pm$ be the open set $(1 + x^1) > 0$ (resp. $(1 + x^1) < 0$) and let $\nabla^\pm$ be the linear symplectic connection on $U^\pm$ defined by Proposition 11.2. Then the curves $x^2 =$ct are geodesics such that $t = 0$ for $x^1 = x_0^1$ and $\frac{\mathrm{d}x^1}{\mathrm{d}t}(0) = B$ ($B < 0$ if $1 + x_0^1 > 0$ and $B > 0$ if $1 + x_0^1 < 0$). One reaches the "boundary" $x^1 = -1$ in a time $\tau = \mp 2/B|1 + x_0^1|^{1/2}$ and the velocity becomes infinite. In particular, the spaces $(U^\pm, \omega_0, \nabla^\pm)$ are geodesically incomplete and cannot be extended.*

The proof is an easy calculation.

Proposition 11.4. *Let $(\mathbb{R}^2, \omega, \nabla)$ be a solution of the field equations and assume $\mathrm{d}\beta \neq 0$. Let U be the open set defined by $U = \{p \mid r(\bar{u}, \bar{u})(p) \neq 0\}$. Then for any $p \in U$, there exists a Darboux chart (V, φ) centered at p with coordinates $x^a (a = 1, 2)$ such that*

(i) $V \subset U$

(ii) $x^2 = \beta|_V - \beta(p)$

(iii) $\mathrm{d}x^1 = \frac{u}{\beta^2 + B}|_V$

(iv) *The Christoffel symbols in this chart have the expression*

$$\nabla_{\partial_1}\partial_1 = -[(x^2 + \beta_p)^2 + B][(x^2 + \beta_p)^2 + 2(x^2 + \beta_p)A - B]\partial_2,$$

$$\nabla_{\partial_1}\partial_2 = \nabla_{\partial_2}\partial_1 = \frac{x^2 + \beta_p}{[(x^2 + \beta_p)^2 + B]}\partial_1,$$

$$\nabla_{\partial_1}\partial_1 = -\frac{x^2 + \beta_p}{[(x^2 + \beta_p)^2 + B]}\partial_2.$$

Proof. If one notices that the 1-form $u/\beta^2 + B$ is closed, the proposition is an easy reformulation of Lemma 5.2.

If one assumes that the constant B is strictly negative ($B = -K$) the formulas above define a preferred connection in the bands of $\mathbb{R}^2$, $|x^2| < K, x^2 > K, x^2 < -K$. The curves $x^1 =$ct are geodesics and using an affine parameter along with them, one sees that one reaches one of the "boundaries" in a finite time and the velocity vanishes at the boundary. So one cannot patch together these various solutions.

If one assumes that the constant $B = 0$, the formulas above define a preferred connection in the upper and in the lower half plane. The curves $x^1 =$ct are geodesics and using an affine parameter, one sees that it takes an infinite time to reach the boundary; the velocity tends to zero when one goes to the boundary. On the other hand, any geodesic along which x^1 is not constant reaches the boundary ($x^2 = 0$) in a finite time and for a finite value of x^1. So one cannot patch together the two solutions.

If one assumes that the constant B is strictly positive, the formulas above define a preferred connection on the whole plane. The geodesic equations are readily integrated and one sees that the connection is geodesically complete.

We thus get the following

Theorem 11.1. *On the plane* $(\mathbb{R}^2, \omega_0)$ *with the standard symplectic structure, there exist a family of preferred connections* ∇ *which are geodesically complete and affinely not equivalent. These are*

(i) $\nabla_0 =$ *the standard flat connection,*
(ii) $\nabla_1^{(k)}$ $=$ *a one parametric family of symmetric connections corresponding to a constant curvature Lorentz metric,*
(iii) $\nabla_2^{(k)}$ $=$ *a one-parameter family of symmetric connections corresponding to a constant negative curvature Riemannian metric,*
(iv) $\nabla_3 =$ *a symmetric connection associated to a coadjoint orbit of the group* $E(2)$,
(v) $\nabla_4 =$ *a symmetric connection associated to a coadjoint orbit of the group* $M(2)$,
(vi) $\nabla_5^{(A,B)}$ $=$*a two parameter family of non-symmetric connections.*

12. Remark for further development

The results in dimension 2 encourage us to look at higher dimension. Let us simply indicate in this direction a construction which given a preferred connection on (M, ω) builds a preferred connection on $(T^*M, d\theta)$ ($\theta =$ Liouville form on the cotangent bundle).

Recall a classical construction due to Yano and Kobayashi [10]. If $\pi : TM \to M$ denotes the tangent bundle to M, one defines 2 lifts of tensor fields on M to tensor fields on TM (of the same type).

The vertical lift is characterized by the following properties:

(i) If S and T are tensor fields on M and if S^v and T^v denote their vertical lifts:

$$(S \otimes T)^v = S^v \otimes T^v.$$

(ii) If $f \in C^\infty(M)$,

$$f^v = \pi^* f.$$

(iii) If $f \in C^\infty(M)$,

$$(\mathrm{d}f)^v = \mathrm{d}(f^v).$$

(iv) If $X \in \Gamma(TM)$ and $f \in C^\infty(M)$,

$$X^v(\mathrm{d}f) = (Xf)^v,$$

where (on the left-hand side) $\mathrm{d}f$ is viewed as a function on TM.

This implies that the vertical lift of a p-form α on M is

$$\alpha^v = \pi^* \alpha.$$

Also if $x^i (i \leq n = \dim M)$ are local coordinates on M and $(x^i, y^i)(i \leq n)$ are the corresponding local coordinates on TM:

$$X^v = \left(\sum_i X^i \frac{\partial}{\partial x^i} \right)^v = \sum_i X^i \frac{\partial}{\partial y^i}.$$

The complete lift of tensor fields is characterized by the following properties:

(i) If S and T are tensor fields on M and S^c, T^c denote their complete lifts:

$$(S \otimes T)^c = S^c \otimes T^v + S^v \otimes T^c.$$

(ii) If $f \in C^\infty(M)$,

$$f^c = \mathrm{d}f.$$

where (as above) $\mathrm{d}f$ is viewed as a function on TM.

(iii) If $f \in C^\infty(M)$,

$$(\mathrm{d}f)^c = \mathrm{d}(f^c).$$

(iv) If $X \in \Gamma(TM)$ and $f \in C^\infty(M)$,

$$X^c(f^c) = (Xf)^c.$$

In particular, using local coordinates on M and TM as above, we have

$$f^c(x, y) = \sum_i \frac{\partial f}{\partial x^i}(x) y^i,$$

$$(\mathrm{d}f)^c(x, y) = \sum_{i,j} \frac{\partial^2 f}{\partial x^j \partial x^i}(x)\, \mathrm{d}x^j y^i + \sum_i \frac{\partial f}{\partial x^i}(x)\, \mathrm{d}y^i,$$

$$X^c(x, y) = \sum_i X^i \frac{\partial}{\partial x^i} + \sum_{i,j} y^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial y^i},$$

$$\alpha^c(x, y) = \sum_{i,j} \frac{\partial \alpha_i}{\partial x^j} y^j\, \mathrm{d}x^i + \sum_i \alpha_i\, \mathrm{d}y^i.$$

Now if ω is a symplectic form on M, ω^c is a 2-form on TM which reads, locally,

$$\omega^c = \frac{1}{2} \sum_{i,j,k} \frac{\partial \omega_{ij}}{\partial x^k} y^k\, \mathrm{d}x^i \wedge \mathrm{d}x^j + \sum_{i,j} \omega_{ij}\, \mathrm{d}y^i \wedge \mathrm{d}x^j$$

and is thus clearly a symplectic form on TM.

The form ω induces a linear isomorphism $\underline{\omega} : TM \to T^*M : X \to i(X)\omega$.

Lemma 12.1. *One has the identity*

$$\omega^c = \underline{\omega}^*(\mathrm{d}\theta).$$

One can lift a symplectic connection ∇ on M to a connection ∇^c on TM and push it forward to T^*M.

The complete lift ∇^c of a connection ∇ on M, is defined by

$$\nabla^c_{X^c}(Y^c) = (\nabla_X Y)^c.$$

In a local chart (x^i, y^i) of TM one has

$$\begin{aligned} &\Gamma^{c\,l}_{ik} = \Gamma^l_{ik}, \quad \Gamma^{c\,l}_{i\underline{k}} = 0, \quad \Gamma^{c\,l}_{\underline{ik}} = 0, \\ &\Gamma^{c\,\underline{p}}_{i\underline{l}} = \Gamma^p_{il}, \quad \Gamma^{c\,\underline{p}}_{ik} = \frac{\partial \Gamma^p_{ik}}{\partial x^j} y^j, \quad \Gamma^{c\,\underline{p}}_{\underline{ij}} = 0, \end{aligned}$$

where barred indices refer to the y^j's coordinates. From these formulas one deduces easily that

$$\begin{aligned} &[X^c, Y^c] = [X, Y]^c, \\ &\nabla^c_{X^c} Y^c - \nabla^c_{Y^c} X^c - [X^c, Y^c] = (\nabla_X Y - \nabla_Y X - [X, Y])^c. \end{aligned}$$

Hence if ∇ is torsion free, so is ∇^c. One also notices that

$$\omega^c(X^c, Y^c) = (\omega(X, Y))^c$$

from which one deduces that ∇ is a symplectic connection for ω, then ∇^c is a symplectic connection for ω^c. Notice finally that

$$R^c(X^c, Y^c)Z^c = (R(X, Y)Z)^c, \qquad r^c(X^c, Y^c) = 2(r(X, Y))^v.$$

Hence

$$\begin{aligned} (\nabla^c_{X^c} r^c)(Y^c, Z^c) &= X^c r^c(Y^c, Z^c) - r^c(\nabla^c_{X^c} Y^c, Z^c) - r^c(Y^c, \nabla^c_{X^c} Z^c) \\ &= 2X^c(r(X, Y))^v - 2(r(\nabla_X Y, Z))^v - 2(r(Y, \nabla_X Z))^v \\ &= 2\,[Xr(Y, Z) - r(\nabla_X Y, Z) - r(Y, \nabla_X Z)]^v \\ &= 2((\nabla_X r)(Y, Z))^v \end{aligned}$$

which proves

Proposition 12.1. *Let (M, ω) be a symplectic manifold and let ∇ be a preferred symplectic connection. Then on $(T^*M, \mathrm{d}\theta)$, the linear connection $\tilde{\nabla}$ defined by*

$$\tilde{\nabla}_X Y = \underline{\omega}_* \nabla^c_{\underline{\omega}_*^{-1} X} \underline{\omega}_*^{-1} Y,$$

where ∇^c denotes the complete lift of ∇ to TM and $\underline{\omega}$ is the linear isomorphism of TM with T^*M, is a symplectic preferred connection.

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